Chromaticity diagram showing cone excitation by stimuli of equal luminance

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In a space where Cartesian coordinates represent the excitations of the three cone types involved in color vision, a plane of constant luminance provides a chromaticity diagram in which excitation of each cone type (at constant luminance) is represented by a linear scale (horizontal or vertical), and in which the center-of-gravity rule applies with weights proportional to luminance.

Because lights that match are similar in their effects upon the three types of cone in an observer's fovea, color and luminance can be specified graphically by a point in three-dimensional space with orthogonal axes showing the excitation of each cone type.[1] In this cone excitation space (Fig. 1),[2] colors may be specified without regard for their intensities by associating each color with a straight line through the origin. The various points on each such line show the cone excitations produced by various intensities at some constant chromaticity. Almost any plane in cone excitation space could serve as a possible chromaticity diagram, in which each color is represented by the point where its line meets the plane. If the plane $R + G + B = 1$ is taken as the chromaticity diagram, colors fall within the obliquely oriented equilateral triangle indicated by dashed lines in Fig. 1(a), the corners of which represent excitation of only one type of cone.

This equilateral chromaticity triangle[3] has the advantage that it incorporates an axis corresponding to each cone type, but it has the drawback that the axes are not orthogonal, each being perpendicular to a side of the triangle. To produce the customary orthogonal coordinates, the equilateral triangle can be projected onto a right-angled triangle, for instance by a vertical projection onto the R, G plane in Fig. 1(a) (an operation similar to that used to generate the standard chromaticity diagram from the CIE color matching functions). But this solution is not completely satisfactory because it deprives the B cones of a direct representation in the diagram, and thereby makes it harder to relate the new diagram to the three-dimensional space from which it was derived. A second, more serious drawback of the equilateral diagram results from the zero, or near zero, contribution of the B cones to luminance.[4] If the axes representing each cone's excitation in the three-dimensional space were scaled to make the plane $R + G + B = 1$ a plane of constant luminance, the values of B would always be very small and the resulting diagram would compress

all colors into a narrow zone close to the baseline in the R, G plane; if B cones make no contribution to luminance, all colors would lie along the R, G baseline and the diagram would fail completely to represent the dimension of chromaticity associated with B cone excitation. Alternatively, if the B axis were scaled up so as to make the B values comparable in magnitude with those of R and G, then the equilateral-triangle diagram would deviate seriously from a plane of constant luminance. An undesirable consequence would be that, for colors of equal luminance, equal distances along a straight line in the diagram would correspond to unequal changes in cone excitation.[5]

Both the above-mentioned drawbacks are avoided in an alternative representation that presupposes no contribution to luminance by B cones. If the units for R and G are chosen to be of equal luminance, equiluminous colors lie in planes within which the sum of R and G is constant. These planes, like the plane $R + G = 1$ shown by dashed lines in Fig. 1(b), are parallel to the B cone axis, no matter how that axis is scaled. Consider the plane $R + G = 1$. This plane is shown in Fig. 2, which may be regarded as the vertical slice indicated by dashed lines in Fig. 1(b), lifted bodily from that three-dimensional space and laid flat on the page. It is oriented so that the point corresponding to the G axis [the point $R = 0$, $G = 1, B = 0$ in Fig. 1(b)] is at the lower left corner and the point corresponding to the R axis is at the lower right corner, with B cone excitation still increasing upward just as it did in Fig. 1(b). The horizontal axis of the new diagram, r, is proportional to R cone excitation (for equiluminous colors). It is simply the fraction of luminance that is attributable to R cones. The fraction of luminance attributable to G cones, g, is, therefore, simply $1 - r$ and is measured by reading the horizontal axis from right to left, as shown by the added scale in Fig. 2. At the far left of the diagram all luminance comes from the G cones; at the far right it all comes from the R cones; and, midway between, the two types of cones contribute
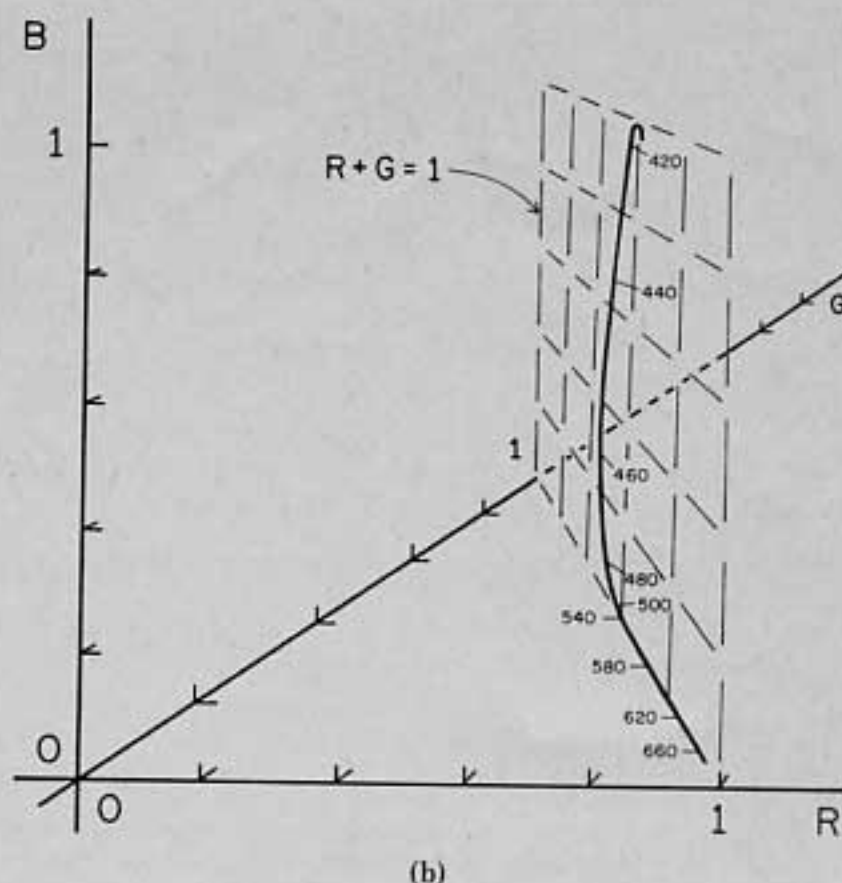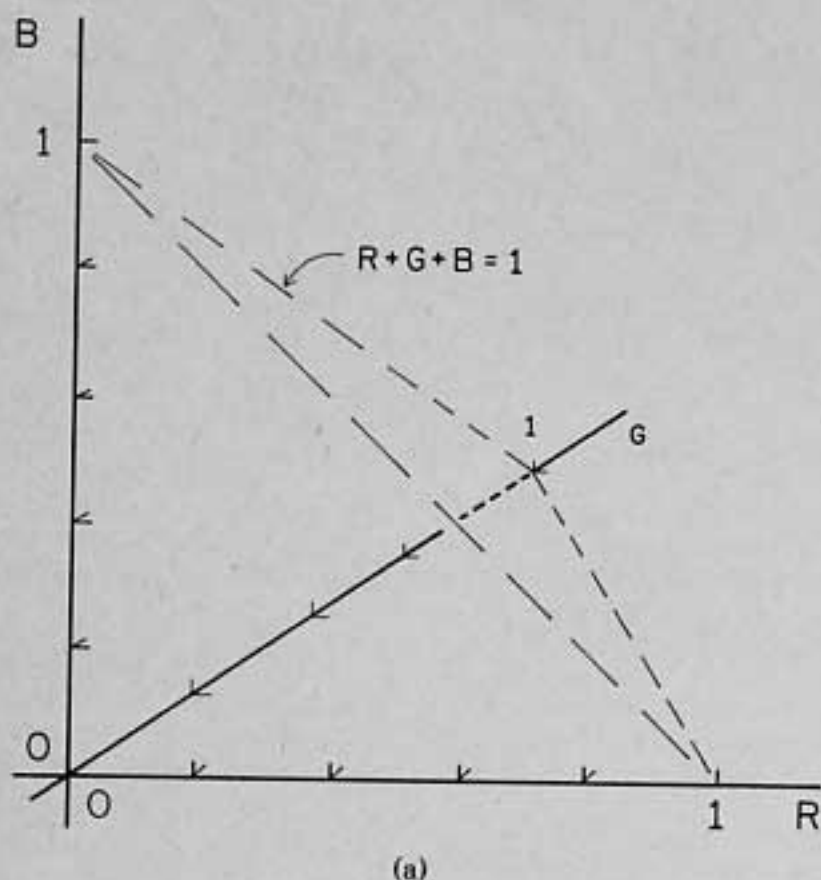
FIG. 1. Cone excitation space. The R and B axes lie in the plane of the paper, and the apparently oblique G axis should be considered as orthogonal to the other two, receding into the distance behind the plane of the paper. In Fig. 1(a) the obliquely oriented equilateral triangle $R + G + B = 1$ is indicated by dashed lines. In Fig. 1(b) dashed lines indicate the plane of constant luminance $R + G = 1$, and the spectrum locus in that plane is shown.

equally. The vertical axis, b, is proportional to B cone excitation for equiluminous colors, and represents B cone excitation per unit luminance. The following equations define the quantities represented in this proposed chromaticity diagram, in terms of the three cone excitations:

$$r = R/(R + G)$$

$$g = G/(R + G) \tag{1}$$

$$b = B/(R + G).$$

Stimuli confused by deuteranopes (who lack G cones) have identical values of B and R and thus lie on lines parallel to the G axis in cone excitation space. For instance, In Fig. 3 the three stimuli a_1, a_2, and a_3 on the confusion line aa_n are confused with each other. The chromaticities of these three confused colors (whatever their intensities) are given by the points at which the indicated lines of constant chromaticity radiating from the origin to a_1, a_2, and a_3 meet the plane of the diagram. Other stimuli on this confusion line are also confused with the three shown, and the lines radiating from the origin that define their chromaticities lie in the plane that contains the confusion line and the origin (and hence also includes the G axis, which is parallel to the confusion line). The intersection of that plane with the plane of the chromaticity diagram is the line Da and this is the locus of chromaticities corresponding to the confusion line aa_n. Similarly, the projections of the other shown confusion lines bb_n, cc_n, and dd_n lie within the indicated planes below that radiate from the G axis to the confusion line in question. The intersections of these radiating planes with the plane of the chromaticity diagram generate the projected confusion lines Db, Dc, and Dd, which converge at the lower left corner of the chromaticity diagram, at D, which is the intersection of the G axis with the

constant luminance plane. Deuteranopes with only B and R cones confuse these colors (at appropriate relative intensities) because the ratio B/R (or equivalently the ratio b/r) is everywhere the same along any of the radiating lines. Protanopic confusion lines similarly radiate from the "R" corner (Fig. 2). There is no B corner in the diagram: tritanopic confusion lines remain vertical and parallel in their projections onto the diagram of Fig. 2. This lack of symmetry in the representation of R, G, and B is not a shortcoming: it is nature's asymmetry, faithfully reflected in the diagram.
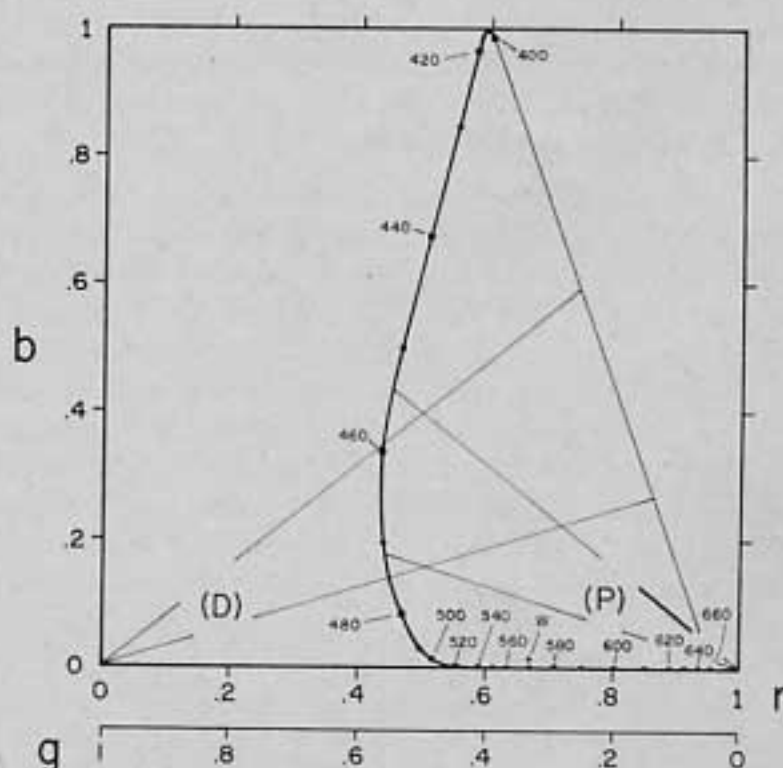

FIG. 2. Proposed (r, b) chromaticity diagram [equivalent to the dashed plane in Fig. 1(b)] showing spectrum locus and confusion lines of protanopes (P) and deuteranopes (D). The point indicated by W is the equal energy white.
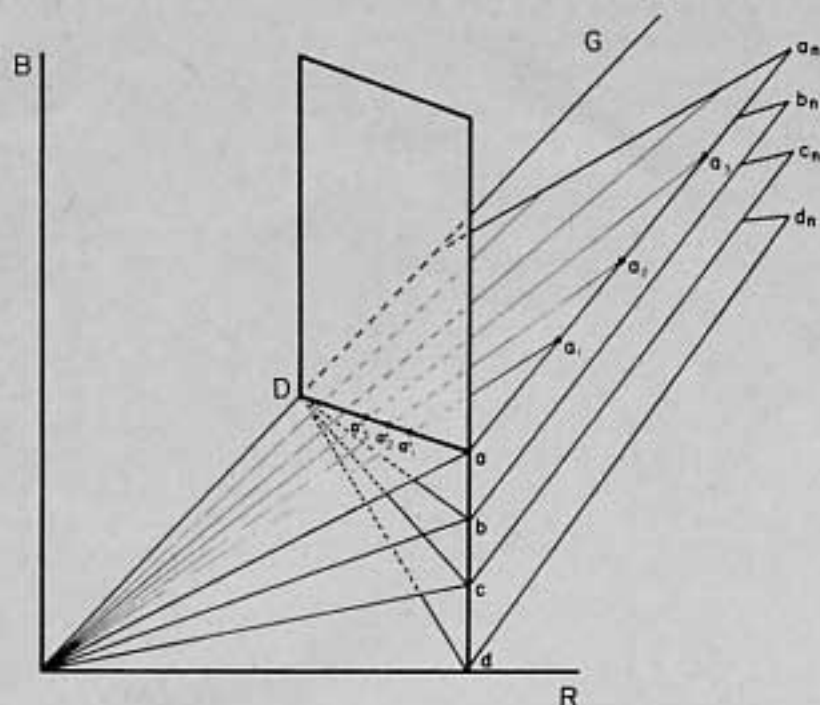
FIG. 3. Projection onto the (r, b) diagram in cone excitation space, illustrated for colors confused by deuteranopes. The confusion lines aa_n, bb_n, cc_n, dd_n are parallel to the G axis. Connections from these four lines to the origin lie in four planes radiating from the G axis as shown. Thus, the projections of the four lines in the plane of the chromaticity diagram are the converging lines Da, Db, Dc, and Dd, which are where the four radiating planes meet the plane of the diagram.

The fact that each vertical line in the (r, b) diagram is a different tritanopic confusion line helps define the relation between the (r, b) diagram and the CIE diagram, where such lines radiate instead from the tritanopic copunctal point. Horizontal lines in the (r, b) diagram represent colors of equal B-cone excitation that are confused by an imaginary dichromat whose long-wavelength spectral sensitivity is the same as Judd's normal luminosity curve; these would project onto the CIE diagram as a fan converging on the point $(1, 0)$.

The chromaticity diagram of Fig. 2 has several advantages over others that have been proposed. (i) It directly represents the excitation of each cone type without the use of oblique coordinates.[6] (ii) Since the constant luminance diagram of Fig. 2 is simply a slice from the cone excitation space of Fig. 1, the relation between the two representations is easily seen by imagining the projection illustrated in Fig. 3. (iii) Because the slice is taken in a plane of constant luminance, equal steps along any straight line in the chromaticity diagram of Fig. 2 (with luminance constant), are equal steps also along a straight line in the cone excitation space, and they correspond to equal changes of excitation for each type of cone. (This is obviously true in the case where stimulus luminance is $R + G = 1$, making the stimuli plotted in the cone excitation space lie in the plane of the chromaticity diagram. For other luminances, it follows either from the fact that planes of constant luminance differ only in scale, or from the equations given above to define the coordinates of the diagram.) (iv) The center-of-gravity rule for color mixture can be applied in an unusually straightforward way, since weights are proportional to luminance. In other words, the mixture of two stimuli having luminance m and n (in any convenient units, so long as they are the same for both) has a luminance $m + n$ and a chromaticity located on the straight line joining the chromaticities of the two components, lying closer to the more luminous component so as to divide the line in the ratio $n:m$. This simple principle should considerably reduce the difficulties presently experienced by beginners in applying the center-of-gravity rule in the CIE chromaticity diagram. In the CIE diagram, the center-of-gravity rule does hold, but the relative weights for the components are given by the *sums of the three tristimulus values* for each component.

To illustrate how the (r, b) diagram reveals relationships that the CIE diagram obscures, consider first this purely colorimetric problem: What luminances of 540 nm and 650 nm primaries will be needed for a match to a 100 td, 590 nm test light? In the CIE system, the ratio of the 540 nm–590 nm distance to the 590 nm–650 nm distance is equal to the ratio of the sums of the tristimulus values for the two primaries in the match. To find the relative luminances, each sum has to be multiplied by the corresponding value of y, which is the ratio of luminance ($\bar{y}$) to the sum of the tristimulus values ($\bar{x} + \bar{y} + \bar{z}$). Finally, from the ratio of the luminances, values summing to 100 td are easily determined. Using the (r, b) diagram, the procedure is simpler and less intuitively opaque. In the diagram (Fig. 2), 590 nm falls 45% of the way along the interval between 540 nm and 650 nm. By the center of gravity rule, the required luminances are therefore in the ratio 55 (of 540 nm) to 45 (of 650 nm), so they must be 55 td and 45 td respectively. When a graphical procedure is insufficiently precise, the relative distances can, of course, be found by evaluating $r_{590} - r_{540}$, etc. All calculations pertaining to the red-green spectral range require only the single quantity r (along with the luminance) for each wavelength involved.

An illustrative problem concerning cone excitations is: By what factor must a light of 540 nm exceed one of 650 nm in luminance, in order to excite equally the R cones? Referring to Eq. (1), R cone excitation is given by the product of r with stimulus luminance, so the required factor is simply r_{650}/r_{540}, or $(0.947)/(0.590) = 1.60$. The same result is obtained in the CIE system by evaluating the expression $[(0.15514 x_{540} + 0.54312 y_{540} - 0.03286 z_{540})/(y_{540})]/[(0.15514 x_{650} + 0.54312 y_{650} - 0.03286 z_{650})/(y_{650})]$.

There is no essential constraint on the scale of the b axis relative to the other two. In Fig. 2 it has been chosen so that the entire spectrum locus will just fit inside a square diagram. When this is done, most colors of everyday interest are situated close to the long-wavelength spectrum locus (see, for example, W, the equal-energy white). An expanded vertical scale would be needed for a precise graphical representation of these colors.

Difference thresholds for discriminations made by the B cones alone are greater in the blue-violet region toward the top of the diagram than near the bottom where the B cones are only weakly excited (see Ref. 6). Therefore, equal vertical distances in the diagram are less noticeable at the top than at the bottom. Similarly, although a given distance anywhere along the horizontal axis represents the same amount of exchange between R and G cone excitations, there is no reason to expect that these will also represent the same degree of perceived color difference.[7]

At constant luminance, varying distances corresponding to chromatic difference thresholds simply reflect the variation of the cone difference thresholds with variation of the test stimulus. In this respect, and also in general, the chart in Fig. 2 successfully portrays physiological relationships that are well disguised in previous chromaticity diagrams. Retaining the

cardinal feature of all linear chromaticity diagrams, that mixtures have chromaticities that plot in accordance with a center of gravity rule on the straight lines connecting the chromaticities of their components, it adds to this a new advantage, that the weights for the different components in color mixture are simply proportional to luminance.[8]

[1]T. N. Cornsweet, *Visual Perception* (Academic, New York, 1970), p. 189.

[2]In this paper the long-wavelength cones (loosely, red-sensitive cones) are referred to as R cones, the midspectral (green-sensitive) cones as G cones and the short wavelength (blue-sensitive) cones as B cones. The quantities R, G, and B are here used to stand for the excitations of the respective cone types by any given stimulus. The corresponding spectral sensitivities are the values of R, G, and B for an equal energy spectrum. The estimated spectral sensitivities for the three cone types used in constructing Figs. 1 and 2, expressed as linear combinations of Judd's 1951 color matching functions, $\bar{x}_j$, $\bar{y}_j$, $\bar{z}_j$, are

$$R_\lambda = 0.15514\,\bar{x}_{j\lambda} + 0.54312\,\bar{y}_{j\lambda} - 0.03286\,\bar{z}_{j\lambda},$$

$$G_\lambda = -0.15514\,\bar{x}_{j\lambda} + 0.45684\,\bar{y}_{j\lambda} + 0.03286\,\bar{z}_{j\lambda},$$

$$B_\lambda = 0.01608\,\bar{z}_{j\lambda}.$$

These are the spectral sensitivities of V. C. Smith and J. Pokorny [Vision Res. 15, 161–172 (1975)]. The coefficient applied to $\bar{z}_{j\lambda}$ is chosen for convenience, as explained in the text in connection with Fig. 2. Diagrams like the one proposed here could be constructed for any set of spectral sensitivities having the property that the sensitivities of the R and G cones summate to give the photopic luminosity function.

[3]The first true chromaticity diagram, suggested by J. Clerk Maxwell, Trans. R. Soc. Edin. 21, 275–298 (1855), was of this form but the equilateral triangle has not been popular in recent times. A. Konig and C. Dieterici, Z. Psych. Physiol. Sinnesorg. 4, 241–347 (1893) and W. A. H. Rushton, Invest. Ophthalmol. 10, 311–322 (1971) have proposed equilateral triangle diagrams based on cone excitations.

[4]For evidence that the B cones make no contribution to luminance, as measured by flicker photometry or by the approximately equivalent minimally distant border method, see Smith and Pokorny (Ref. 2), B. W. Tansley and R. M. Boynton, Vision Res. 18, 683–697 (1978), B. W. Tansley and R. J. Glushko, Vision Res. 18, 699–706 (1978).

[5]This drawback is characteristic of the CIE chromaticity diagram and all other chromaticity diagrams that have previously been proposed.

[6]A nonlinear transformation of a similar diagram, sharing this advantage, was introduced by R. W. Rockeck [*The Vertebrate Retina* (W. H. Freeman, San Francisco, 1973), p. 737] for the analysis of color discrimination.

[7]There is in fact reason to expect otherwise [see, for example, Fig. 6.57 on p. 558 of G. Wyszecki and W. S. Stiles, *Color Science*, (Wiley, New York, 1967)].

[8]Working copies of the chromaticity diagram, and tables of the chromaticity coordinates (r_λ, b_λ) for spectral lights, are available from the authors.